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Some moment properties of skew-symmetric circular distributions

Summary - Many popular circular distributions, including the most commonly used von Mises distribution, are typically symmetric around a modal direction. To enlarge this class, the authors recently discussed families of skew-symmetric distributions, applying this idea to the von Mises model as an important special case. This paper explores further properties of these skew-symmetric families and their moments.

Key Words - Asymmetric circular distributions; Trigonometric moments; Von Mises distribution.

1. INTRODUCTION

Umbach and Jammalamadaka (2009) introduced a class of circular skew symmetric distributions, along the lines of Azzalini and Capitanio (2003), which introduced a more general form of the skew-symmetric distributions originally introduced by Azzalini (1985). In this work, we consider some monotonicity properties of the trigonometric moments of such distributions. Recall that a circular pdf is a non-negative periodic function which integrates to one over intervals of length 2π (see for instance Section 2.1 of Jammalamadaka and SenGupta (2001)). Functions $f(\cdot)$ which satisfy

$$f(\theta + 2\pi) = f(\theta) \quad \text{for all } \theta \tag{1}$$

will be simply referred to as *periodic*. The actual period, possibly less than 2π, plays no role in what follows.

In particular, the authors showed that if

 (i) f and g are circular densities which are symmetric about 0,
 (ii) $G(\theta) = \int_{-\pi}^{\theta} g(\alpha)\,d\alpha$, and
(iii) w is odd and periodic with $|w(\theta)| \le \pi$ for all θ,

then

$$f_\mu(\theta) = 2 f(\theta - \mu) G(w(\theta - \mu)) \tag{2}$$

is a circular density for each real number μ. The choice of $-\pi$ for the lower limit of the integral defining G is because we will consider the support of the distributions herein as being over $\theta \in [-\pi, \pi)$ throughout. The resulting density is typically asymmetric. In what follows we study some properties of the trigonometric moments of these distributions, particularly aspects of monotonicity.

2. BASIC RESULTS

The results in this paper follow primarily from the lemmas that we state and prove in this section. They establish useful properties of expected values of even ($h(-\theta) = h(\theta)$ for all θ) and odd ($h(-\theta) = -h(\theta)$ for all θ) functions of random variables which have skew symmetric circular distributions defined in Equation (2).

In what follows, h_o is an odd function. It is convenient to define $D_o^+ = \{\theta | -\pi \leq \theta < \pi \text{ and } h_o(\theta) > 0\}$ and $D_o^- = \{\theta | -\pi \leq \theta < \pi \text{ and } h_o(\theta) < 0\}$. In addition to (i), (ii), and (iii), the lemmas are based on the following hypotheses:

(iv) For the odd function h_o, and $\theta \in D_o^+$, $w(\theta) \geq 0$ and for $\theta \in D_o^-$, $w(\theta) \leq 0$.
 (v) For the odd function h_o, and $\theta \in D_o^+$, $w(\theta) \leq 0$ and for $\theta \in D_o^-$, $w(\theta) \geq 0$.
(vi) h_e is even and periodic.

Conditions (i), (ii), and (iii) insure that f_μ in (2) is indeed a circular density. The density labeled f_0 in the three lemmas below is such a density with $\mu = 0$. The other conditions insure various monotonicity properties as described in the lemmas. The lemmas are presented together with, insofar as possible, a unified proof. The linear version of Lemma 2 was established in Azzalini and Capitanio (2003).

Lemma 1. *Hypotheses* (i), (ii), (iii), *and* (iv) *imply that*

$$0 = \int_{-\pi}^{\pi} h_o(\theta) f(\theta) d\theta \leq \int_{-\pi}^{\pi} h_o(\theta) f_0(\theta) d\theta \leq 2 \int_{D_o^+} h_o(\theta) f(\theta) d\theta.$$

Lemma 1a. *Hypotheses* (i), (ii), (iii), *and* (v) *imply that*

$$-2 \int_{D_o^+} h_o(\theta) f(\theta) d\theta \leq \int_{-\pi}^{\pi} h_o(\theta) f_0(\theta) d\theta \leq \int_{-\pi}^{\pi} h_o(\theta) f(\theta) d\theta = 0.$$

Lemma 2. *Hypotheses* (i), (ii), (iii), *and* (vi) *imply that*

$$\int_{-\pi}^{\pi} h_e(\theta) f(\theta) d\theta = \int_{-\pi}^{\pi} h_e(\theta) f_0(\theta) d\theta = \int_{0}^{\pi} 2 h_e(\theta) f(\theta) d\theta.$$

Proof. The symmetry of f and the fact that h_o is odd implies $\int_{-\pi}^{\pi} h_o(\theta) f(\theta) d\theta = 0$. Now, note that

$$\int_{-\pi}^{\pi} h_o(\theta) f_0(\theta) d\theta$$

$$= \int_{D_o^+} h_o(\theta) 2f(\theta) G(w(\theta)) d\theta + \int_{D_o^-} h_o(\theta) 2 f(\theta) G(w(\theta)) d\theta$$

$$= \int_{D_o^+} h_o(\theta) 2 f(\theta) G(w(\theta)) d\theta - \int_{D_o^+} h_o(\gamma) 2f(\gamma) (1 - G(w(\gamma))) d\gamma \quad (3)$$

$$= 2 \int_{D_o^+} h_o(\theta) 2 f(\theta) G(w(\theta)) d\theta - \int_{D_o^+} h_o(\theta) 2 f(\theta) d\theta \quad (4)$$

$$\leq 2 \int_{D_o^+} h_o(\theta) 2 f(\theta) d\theta - \int_{D_o^+} h_o(\theta) 2 f(\theta) d\theta \quad (5)$$

$$= 2 \int_{D_o^+} h_o(\theta) f(\theta) d\theta.$$

Note that (3) follows from the symmetry of g, which necessitates $G(w(\theta)) = 1 - G(-w(\theta))$ for all θ, and the fact that w is odd, followed by the substitution $\gamma = -\theta$ in the second integral. The inequality in (5) is established by noting that $G(w(\theta)) \leq 1$ for all θ.

One establishes that

$$\int_{-\pi}^{\pi} h_o(\theta) f_0(\theta) d\theta \geq 0$$

by using (iv) to see that for $\theta \in D_o^+$, we have $w(\theta) \geq 0$. Thus (i) and (ii) yield $G(w(\theta)) \geq 1/2$ for these values of θ and thus $2 G(w(\theta)) \geq 1$. The result then follows from (4), completing the proof of Lemma 1.

For the proof of Lemma 1a, we note that the above development holds through (4). Noting that the first integral of (4) is positive, we see that

$$\int_{-\pi}^{\pi} h_o(\theta) f_0(\theta) d\theta \geq -2 \int_{D_o^+} h_o(\theta) f(\theta) d\theta.$$

The proof is completed by noting that (v) implies that $2 G(w(\theta)) \leq 1$ for $\theta \in D_o^+$, and thus we see that (4) is less than or equal to 0.

For Lemma 2, note that the symmetry of f and h_e about 0 implies

$$\int_{-\pi}^{\pi} h_e(\theta) f(\theta) d\theta = \int_0^{\pi} 2h_e(\theta) f(\theta) d\theta.$$

Now

$$\int_{-\pi}^{\pi} h_e(\theta) \, f_0(\theta) \, d\theta$$

$$= \int_0^{\pi} h_e(\theta) \, 2f(\theta) \, G(w(\theta)) \, d\theta + \int_{-\pi}^0 h_e(\theta) \, 2 \, f(\theta) \, G(w(\theta)) \, d\theta$$

$$= \int_0^{\pi} h_e(\theta) \, 2 \, f(\theta) \, G(w(\theta)) d\theta + \int_0^{\pi} h_e(\theta) \, 2 \, f(\theta)(1 - G(w(\theta)) d\theta$$

$$= \int_0^{\pi} 2 \, h_e(\theta) \, f(\theta) \, d\theta,$$

which completes the proof. □

3.   TRIGONOMETRIC MOMENTS

For a circular random variable Θ, the pth trigonometric moment φ_p, is defined by

$$\varphi_p = \mathrm{E}(e^{i \, p \, \Theta}) = \mathrm{E}(\cos p \, \Theta) + i \, \mathrm{E}(\sin p \, \Theta) = \alpha_p + i \, \beta_p.$$

If we define the function $Atan$ by

$$Atan(x, y) = \begin{cases} \arctan(y/x) & \text{for } x > 0 \\ \arctan(y/x) + \pi & \text{for } x < 0, \ y > 0 \\ \arctan(y/x) - \pi & \text{for } x < 0, \ y \le 0 \\ \pi/2 & \text{for } x = 0, \ y > 0 \\ -\pi/2 & \text{for } x = 0, \ y < 0, \end{cases}$$

then $Atan(\alpha_1, \beta_1)$ gives the mean direction of the distribution. The length of φ_p is $\rho_p = \sqrt{\alpha_p^2 + \beta_p^2}$. The value of ρ_1 is used as a basic measure of concentration around the mean (see again Section 2.1 of Jammalamadaka and SenGupta (2001)).

In this section, we consider trigonometric moments of three distributions related to (2) with w having the additional property that $w(\theta) \ge 0$ for $0 \le \theta \le \pi$. With $\mu = 0$, the trigonometric moments and related quantities of the kernel, f, will be denoted with a superscript of ^{o} as φ_p^o, for example. These quantities will have a superscript of $'$ for the distribution f_0 of (2) (with $\mu = 0$) and will have a superscript of * for the folded distribution $2 f(\theta)$ for $0 \le \theta < \pi$.

Since $h_o(\theta) = \sin \theta$ is an odd function with $D_o^+ = (0, \pi)$, Lemma 1 can be applied to conclude that

$$0 = \beta_1^o \le \beta_1' \le \beta_1^*. \tag{6}$$

While $h_o(\theta) = \sin p\,\theta$ is also an odd function, we cannot establish results similar to (6) for arbitrary p because D_o^+ varies with p. However, since $h_e(\theta) = \cos p\,\theta$ is an even function, Lemma 2 allows us to conclude that $\alpha_p^o = \alpha_p' = \alpha_p^*$ for all p. We will denote the common value simply by α_p. These results immediately yield

$$\rho_1^o \le \rho_1' \le \rho_1^*.$$

Thus, we see that the concentration of the three distributions is greatest for the kernel, smallest for the folded distribution, and is between these extremes for the skew symmetric distribution. These results parallel those that have been established for the variance in the linear case by Umbach (2006). Using (6), we see that $\mu^o = 0$, of course. Now, $\mu' = Atan(\alpha_1, \beta_1')$ and $\mu^* = Atan(\alpha_1, \beta_1^*)$. Since arctan is increasing, we obtain the monotonicity result for the mean direction that

$$0 = \mu^o \le \mu' \le \mu^* < \pi.$$

If in addition $\alpha_1^o = 0$, then we have an interesting situation because we then also have $\alpha_1' = \alpha_1^* = 0$. In this case, if the mean of the skewed distribution exists, it will be $\mu' = Atan(0, \beta_1') = \pi/2$. A condition that guarantees $\alpha_1^o = 0$ is that the kernel f is symmetric about $\pi/2$ in addition to the symmetry about 0. In this case, we have $f(\theta) = f(\pi - \theta)$ for $\pi/2 \le \theta \le \pi$ in addition to $f(\theta) = f(-\theta)$. Thus, we see that

$$
\begin{aligned}
\alpha_1^o &= \int_{-\pi}^{\pi} \cos(\theta)\, f(\theta)\, d\theta \\
&= \int_{-\pi}^{0} \cos(\theta)\, f(\theta)\, d\theta + \int_{0}^{\pi} \cos(\theta)\, f(\theta)\, d\theta \\
&= 2 \int_{0}^{\pi} \cos(\theta)\, f(\theta)\, d\theta \\
&= 2 \int_{0}^{\pi/2} \cos(\theta)\, f(\theta)\, d\theta + 2 \int_{\pi/2}^{\pi} \cos(\theta)\, f(\theta)\, d\theta \\
&= 2 \int_{0}^{\pi/2} \cos(\theta)\, f(\theta)\, d\theta - 2 \int_{0}^{\pi/2} \cos(\gamma)\, f(\gamma)\, d\gamma = 0,
\end{aligned}
$$

using the substitution $\gamma = \pi - \theta$. By (6), the mean direction will exist if $\beta_1' > 0$. Using (4) with $h(\theta) = \sin \theta$ we see that

$$\beta_1' = 2 \int_{0}^{\pi} \sin(\theta)\, f(\theta)\, (G(w(\theta)) - 1)\, d\theta,$$

which will be positive if there exists an open interval in $(0, \pi)$ on which both $w(\theta) > 0$ and $f(\theta) > 0$.

4. PARAMETRIC FAMILIES

By incorporating one or more parameters in the definition of w, we can introduce families of distributions. Judicious choices of such parameters will lead to the original distribution as a member of the family with sufficient variation about it to produce a useful family for modeling purposes. For example, if $w(\theta) = \lambda \pi \sin(k\theta)$ for some integer k, then we get a family of distributions from (2) given by

$$2 f(\theta - \mu) G(\lambda \pi \sin(k(\theta - \mu)))$$

for $-1 \leq \lambda \leq 1$. The choice of $\lambda = 0$ produces the original distribution because we must have $G(0) = 1/2$.

Quite generally, we will consider the family of distributions $\{f'_\lambda\}$ for $-1 \leq \lambda \leq 1$ given by

$$f'_\lambda(\theta) = 2 f(\theta) G(\lambda w(\theta)) \tag{7}$$

where f, G, and w satisfy (i), (ii), and (iii) of Section 2. Let $\varphi_p(\lambda) = \alpha_p(\lambda) + i\beta_p(\lambda)$ be the pth trigonometric moment of f'_λ. However, since each f'_λ has the form (2), we see that

$$\alpha_p^o = \alpha_p(\lambda) = \alpha_p^*.$$

Thus, $\alpha_p(\lambda)$ is constant in λ. So we suppress the variable λ and simply write α_p for the common value. Under these conditions, we obtain the monotonicity result that $\varphi_p(\lambda) = \alpha_1 + i\beta_1(\lambda)$ with $\beta_1(\lambda)$ increasing with λ, as is shown in Theorem 1.

Theorem 1. *Suppose that conditions* (i), (ii), *and* (iii) *hold and f'_λ is given by* (7) *with $w(\theta) \geq 0$ for $0 \leq \theta < \pi$ and $-1 \leq \lambda \leq 1$.*

Then

$$\varphi_1(\lambda) = \int_{-\pi}^{\pi} e^{i\theta} f'_\lambda(\theta) \, d\theta = \alpha_1 + i\,\beta_1(\lambda),$$

where $\beta_1(\lambda)$ is an increasing function of λ.

Proof. Note that

$$
\begin{aligned}
\beta_1(\lambda) &= \int_{-\pi}^{\pi} \sin(\theta) f(\theta) 2G(\lambda w(\theta)) d\theta \\
&= \int_{0}^{\pi} \sin(\theta) f(\theta) 2G(\lambda w(\theta)) d\theta + \int_{-\pi}^{0} \sin(\theta) f(\theta) 2G(\lambda w(\theta)) d\theta \\
&= \int_{0}^{\pi} \sin(\theta) f(\theta) 2G(\lambda w(\theta)) d\theta - \int_{0}^{\pi} \sin(\theta) f(\theta) 2G(-\lambda w(\theta)) d\theta \\
&= \int_{0}^{\pi} \sin(\theta) f(\theta) 2[G(\lambda w(\theta)) - G(-\lambda w(\theta))] d\theta.
\end{aligned}
\tag{8}
$$

Since $w(\theta) \geq 0$ for $0 \leq \theta \leq \pi$, we find that $G(\lambda\, w(\theta)) - G(-\lambda\, w(\theta))$ is an increasing function of λ for each θ, and thus (8) is an increasing function of λ. Since $\lambda\, w(\theta)$ satisfies (iii) of Lemma 2 for each λ and $\cos\theta$ is even, we establish that

$$\int_{-\pi}^{\pi} \cos(\theta) f_\lambda'(\theta)\, d\theta$$

does not depend on λ, which thus yields the result.

If one does not stipulate that either $w(\theta) \geq 0$ for $0 \leq \theta \leq \pi$ or $w(\theta) \leq 0$ for $0 \leq \theta \leq \pi$, then monotonicity results will be difficult to establish. This can be seen from (8). Monotonicity results will be quite messy if one has $G(\lambda\, w(\theta)) - G(-\lambda\, w(\theta))$ increasing for some values of θ and decreasing for others. Note also that $\beta_1(\lambda)$ is a decreasing function of λ if the sign condition on w is reversed.

Note that one may use $\pi \sin\theta$ for $w(\theta)$ in (iii). If this is combined with G from the uniform distribution over $[-\pi, \pi)$, i.e. $G(\theta) = (\pi + \theta)/2\pi$, we see that for symmetric f that $f_\lambda' = f(\theta)(1 + \lambda \sin\theta)$ is a skew symmetric circular distribution. In addition, $\beta_1(\lambda)$ can be expressed as $2\lambda \int_0^\pi \sin^2\theta f(\theta)\, d\theta$. $\qquad\square$

5. A SKEW-SYMMETRIC VON MISES DISTRIBUTION

The density of the skewed von Mises distribution, as presented in Umbach and Jammalamadaka (2009), namely

$$v_\lambda(\theta; \kappa) = \frac{e^{\kappa\, \cos\theta}}{2\pi\, I_0(\kappa)}\, (1 + \lambda \sin\theta) \tag{9}$$

for $-1 \leq \lambda \leq 1$, has a density of the form described at the very end of the last section. (Note that $I_p(\kappa)$ is the modified Bessel function of the first kind of order p.) Following Umbach and Jammalamadaka (2009), we define $A_p(\kappa) = I_p(\kappa)/I_0(\kappa)$. We note that $\alpha_1 = A_1(\kappa)$ and

$$\beta_1(\lambda) = 2\lambda \int_0^\pi \sin^2(\theta) \frac{e^{\kappa\, \cos\theta}}{2\pi\, I_0(\kappa)} d\theta = (\lambda/\kappa)\, A_1(\kappa).$$

More generally, we find that the pth trigonometric moment of this distribution has components $\alpha_p = A_p(\kappa)$ and

$$\begin{aligned}
\beta_p(\lambda) &= \int_{-\pi}^{\pi} \sin(p\,\theta) v_\lambda(\theta; \kappa) d\theta \\
&= \int_{-\pi}^{\pi} \sin(p\,\theta) v_0(\theta; \kappa) d\theta + \lambda \int_{-\pi}^{\pi} \sin(\theta) \sin(p\,\theta) v_0(\theta; \kappa) d\theta \\
&= \lambda\, (p/\kappa)\, A_p(\kappa).
\end{aligned} \tag{10}$$

We establish (10) by an integration by parts on the second integral with $u = \sin p\,\theta$ and $v = -e^{\kappa\,\cos\theta}/\kappa$. Thus, we see that the pth trigonometric moment of (9) has a straightforward form based on the modified Bessel functions of orders p and 0. The pth trigonometric moment of the von Mises distribution is $A_p(\kappa)$ when $\mu = 0$. This allows us to see that (10) is consistent with the expression for the pth trigonometric moment given in Theorem 2 of Jammalamadaka and Umbach (2009) which has the form $A_p(\kappa) + i(\lambda/2)(A_{p-1}(\kappa) - A_{p+1}(\kappa))$ when $\mu = 0$. The equivalence is established by noting that $I_{p-1}(\kappa) - I_{p+1}(\kappa) = (2\,p/\kappa)I_p(\kappa)$ as is shown, for example, in Relton (1965).

6. SUMMARY

We see that the trigonometric moments (particularly the first) of skew-symmetric circular distributions as given in (2) have identifiable monotonicity properties. These parallel the linear monotonicity results of Umbach (2006 and 2008).

As shown at the end of Section 4, one can form a skew-symmetric circular density from a symmetric one, say f, by $f'_\lambda = f(\theta)(1 + \lambda \sin\theta)$ for $-\pi \leq \theta \leq \pi$. In particular, if this is used to skew the von Mises distribution, we find that the pth trigonometric moment has the form

$$\varphi'_p = A_p(\kappa)(1 + i\,\lambda p/\kappa),$$

where A_p is a ratio of modified Bessel functions as defined in Section 5. This distribution is studied in some detail in Abe and Pewsey (2009).

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